Libration and Cassini states of celestial bodies with a viscous spherical core: A new analytical framework

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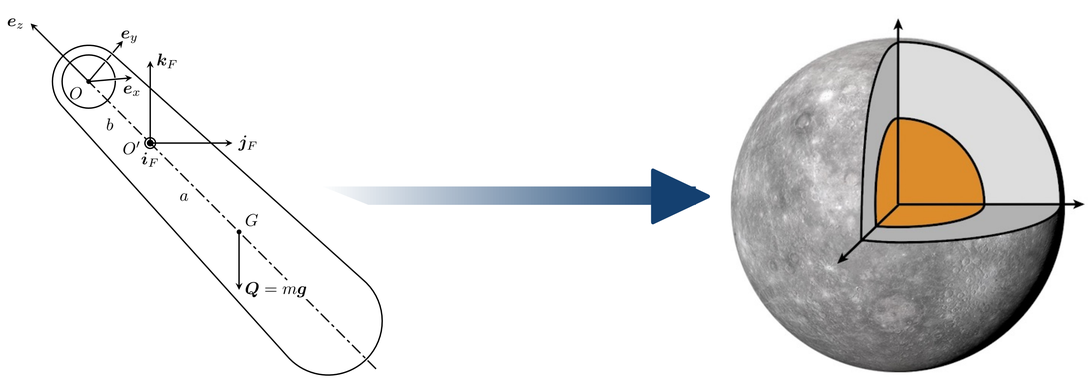
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Introduction

Analytical models describing the rotation of celestial bodies with a viscous fluid core commonly assume a simple motion for the fluid - a solution originally derived by Hough[1] and Poincaré[2] for a perfect (inviscid) fluid. In these models, referred to CMF models, viscosity is subsequently introduced through a friction torque at the core-mantle interface, which depends on the differential rotation between the core and the mantle, such that $\mathbf{\Gamma} = k_c(\boldsymbol{\omega}_c - \boldsymbol{\omega}_m)$.

We propose a new analytical model in which the fluid core motion is described more realistically through a solution of the linearized Navier-Stokes equations. Our methodology is inspired by the work of Ievleva[3, 4], who studied the motion of a pendulum containing a spherical cavity filled with an incompressible viscous fluid. We will examine the effects of this change on planetary rotational dynamics.

From Ievleva's pendulum to celestial mechanics



Hypotheses

- Rigid axisymetric ellipsoidal pendulum and spherical cavity
- The pendulum, which is similar to a top, undergoes small oscillations in addition to its proper rotation

Two equations : linearized Navier–Stokes and angular momentum theorem

$$\frac{\partial \boldsymbol{u}}{\partial t} + 2\omega_z\, \boldsymbol{e}_z \times \boldsymbol{u} + \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \boldsymbol{r} = \nabla P + \nu \boldsymbol{\Delta u} \qquad \frac{\mathrm{d}\,(\boldsymbol{L}_B + \boldsymbol{\ell})}{\mathrm{d}t} = \boldsymbol{M}_G$$

- $\boldsymbol{u}$: relative velocity field of the fluid
- $\boldsymbol{\omega}$: angular velocity of the solid
- P & ν : effective pressure field & kinematic viscosity
- $\boldsymbol{L}_B$: angular momentum of the entire body as it was an entire solid body
- $\boldsymbol{\ell}$: relative angular momentum contribution from the fluid core
- $\boldsymbol{M}_G$: Gravitational torque

Projection onto vector spherical harmonics

$$\text{Radial} : \boldsymbol{Y}_{\ell m} = Y_{\ell m}\, \boldsymbol{e}_r \qquad \text{Toroidal} : \boldsymbol{\Psi}_{\ell m} = r\nabla Y_{\ell m} \qquad \text{Poloidal} : \boldsymbol{\Phi}_{\ell m} = \boldsymbol{r} \times \nabla Y_{\ell m}$$

$$\boldsymbol{u} = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \Big(f_{\ell m}(r,t)\boldsymbol{Y}_{\ell m}(\theta,\phi) + g_{\ell m}(r,t)\boldsymbol{\Psi}_{\ell m}(\theta,\phi) + h_{\ell m}(r,t)\boldsymbol{\Phi}_{\ell m}(\theta,\phi) \Big) \qquad P = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} F_{\ell m}(r,t) Y_{\ell m}(\theta,\phi)$$

Problem : recursive coupling

f_{2m} and g_{2m} depend on h_{1m} and h_{3m}; by recurrence, h_{1m} depends on *all odd-degree* $h_{\ell m}$:

$$h_{1m} \longleftrightarrow \{f_{2m}, g_{2m}\} \longleftrightarrow \{h_{1m}, h_{3m}, h_{5m}, \dots\}$$

Ievleva's hypothesis

$$\boxed{g_{2m} = 0 \implies h_{1m} \text{ decoupled from the hierarchy}}$$

This cancellation breaks the recursive chain: h_{1m} is now accessible independently of $h_{3m}, h_{5m}, \dots$

The mode $\ell = 1$ of the toroidal projection of the Navier-Stokes equation

$$\frac{\partial h_{1,-1}}{\partial t} - \sqrt{\tfrac{2\pi}{3}}\, r \frac{\mathrm{d}\,(\omega_x + \mathrm{i}\omega_y)}{\mathrm{d}t} + \mathrm{i}\omega_z h_{1,-1} = \nu \left[\frac{\partial^2 h_{1,-1}}{\partial r^2} + \frac{2}{r}\frac{\partial h_{1,-1}}{\partial r} - \frac{2}{r^2} h_{1,-1} \right] \qquad \frac{\partial h_{1,0}}{\partial t} - \sqrt{\tfrac{4\pi}{3}}\, r \frac{\mathrm{d}\omega_z}{\mathrm{d}t} = \nu \left[\frac{\partial^2 h_{1,0}}{\partial r^2} + \frac{2}{r}\frac{\partial h_{1,0}}{\partial r} - \frac{2}{r^2} h_{1,0} \right]$$

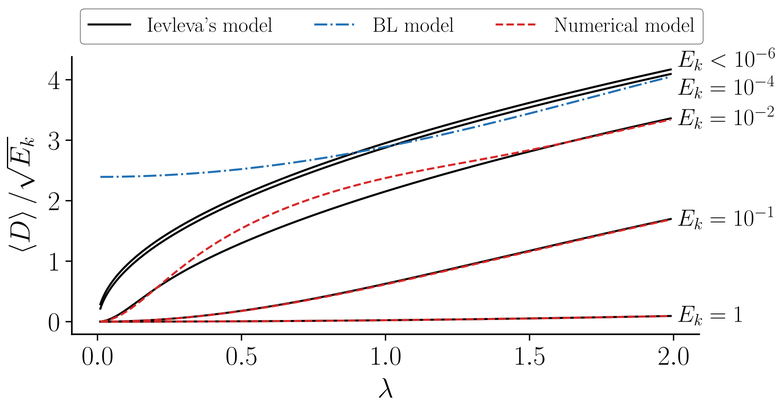
Why this matters

The fluid angular momentum reduces *entirely* to the $\ell = 1$ toroidal mode:

$$\ell_x + \mathrm{i}\,\ell_y = -4\sqrt{\tfrac{2\pi}{3}}\,\rho \int_0^{R_c} h_{1,-1}\, r^3 \mathrm{d}r \qquad \ell_z = -2\sqrt{\tfrac{4\pi}{3}}\,\rho \int_0^{R_c} h_{1,0}\, r^3 \mathrm{d}r$$

The angular momentum of the relative motion depends *only* on the two toroidal components of order $\ell = 1$

Numerically tested model

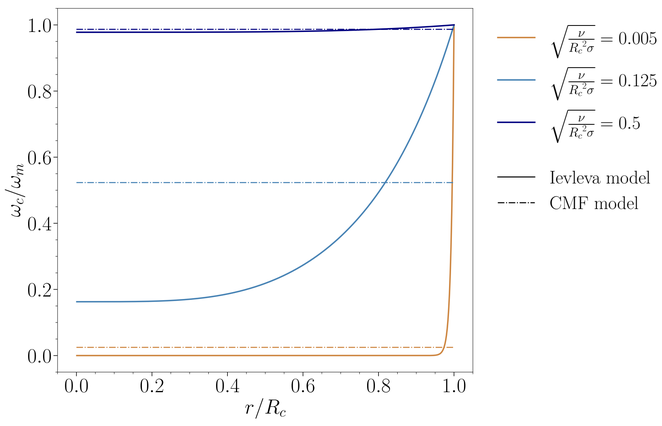


Dissipation, averaged over one libration period and normalized by the square root of the Ekman number, as a function of $\lambda = \sigma/\bar{\omega}_z$ – the ratio between the forcing frequency σ and the mantle's natural angular frequency $\bar{\omega}_z$ – for different Ekman numbers. For $E_k \geq 10^{-2}$, Ievleva's solution agrees very well with the numerical one, unlike the boundary layer model (BL)[7]. For $E_k \leq 10^{-4}$, it remains close to the BL solution for $\lambda > 0.5$, where the latter agrees with the numerical model.

Ievleva's solution appears to remain valid for all Ekman numbers

Comparison with the CMF Model

Angular velocity of the fluid core



Angular velocity of the fluid core as a function of the radial coordinate for different viscosities. The CMF model assumes a uniform angular velocity within the core[8], and introduces a discontinuity at the core–mantle interface. By contrast, in Ievleva's model, the fluid follows the mantle motion at the boundary.

The connection between the CMF model and Ievleva's model

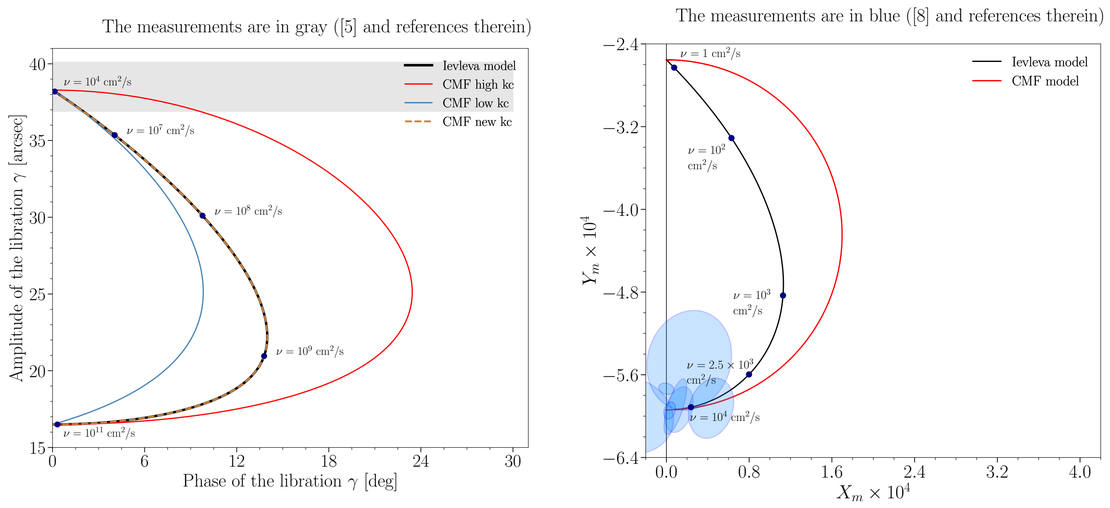
$$k_c = \mathrm{i}\sigma C_c \frac{10\Sigma\left(\sqrt{\nu/({R_c}^2\sigma)}\right)}{1 - 10\Sigma\left(\sqrt{\nu/({R_c}^2\sigma)}\right)} \quad \text{for all viscosity}$$

Σ is a function of the spherical Bessel functions

$$k_c = 35\frac{C_c}{{R_c}^2}\nu \quad \text{at high viscosity} \qquad k_c = 5\frac{C_c}{R_c}\sqrt{\mathrm{i}\sigma\nu} \quad \text{at low viscosity}$$

The parameterization of k_c makes the model straightforward to implement

Application to Mercury



By combining the constraints obtained from both the orientation of the body (right figure) and the libration amplitude (left figure), we derive a viscosity range, for this model, extending from $\sim 10^4\,\mathrm{cm^2\,s^{-1}}$ to $2 \times 10^6\,\mathrm{cm^2\,s^{-1}}$ for Mercury.

This model provides a range of solutions that fits both observations

Key results and future research directions

- The formalism is robust **regardless of the core viscosity regime**, from nearly inviscid to strongly dissipative fluids
- Solutions keep the **same analytical structure** as earlier models, making the approach **directly usable**
- A **single viscosity** simultaneously reproduces the libration and the Cassini state
- A future introduction of **electromagnetic friction**, along with a **new core geometry**, may **change the viscosity** value predicted by the model presented here
- Extend Ievleva's method to include a **solid inner core**

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